What is Algebraic Thinking?

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v2017.10.25

On the surface, algebraic thinking is simply about working with letters as if they were numbers. But the deep idea of algebra is *abstraction*, expressed by this clever symbolic notation.

What is the big idea in algebra? Using letters that stand for a multitude of things – most notably numbers. This way we can write less and say a lot more.

For instance, we can write

$$(a+b)(a-b) = a^2 - b^2,$$

meaning that

$$(1+2)(1-2) = 1^2 - 2^2,$$

which is true since $3 \cdot (-1) = 1 - 4$. It also means that

$$(\pi + \sqrt{3})(\pi - \sqrt{3}) = \pi^2 - 3,$$

and

$$\left(\frac{1}{7} + \frac{2}{5}\right) \cdot \left(\frac{1}{7} - \frac{2}{5}\right) = \frac{1}{49} - \frac{4}{25},$$

and so on. Indeed, this widely known algebraic identity represents infinitely many true statements, infinitely many valid equations of the same form. We can put any pair of numbers into a and b, and there are infinitely many numbers. The letters manage to compress this infinity of possible choices down into a single statement. Extremely economical notation.

By numbers, let's say, we mean the *real numbers* $\mathbb{R}$, that are more than sufficient for our everyday counting and measuring needs.

Is algebra just mere notation then? In a way yes, but notation is much more important than one would think. It is not just for writing down what we already know. It can also be used for discovering new knowledge. Suppose for a moment that we do not know the above algebraic identity. Let's start from the left side,

$$(a+b)(a-b).$$

We can cross-multiply and we get

$$a^2 + ba - ab - b^2.$$

Now, $ba = ab$ since numbers multiplied together yield the same result regardless the order in which we do the multiplication. Therefore we can write

$$a^2 + ab - ab - b^2.$$

We still do not know what exactly ab is, since we did not specify a and b. But certainly, adding it then taking it away is the same as not doing anything, so we end up with

$$a^2 - b^2,$$

thus we establish the algebraic identity. Since we did not say anything about a or b, just used the fact that numbers can be added and multiplied together, therefore $(a+b)(a-b) = a^2 - b^2$ is true for all particular choices of numbers. We have just proved infinitely many statements with a few steps. This is the power of algebraic thinking.

Abstraction works the same way in everyday life. We can talk about more things at the same time by saying less about their specific properties. For instance, apples come in different types and sizes. They can be red, green, or yellow. By simply saying 'apple' we can refer to all of them. When someone says 'I like apples', there is no need to ask 'Do you like this one? and that one?' when going through a bag of apples. In a recipe for apple pie, it is specified how many apples do we need. But it does not matter whether we get apples from the garden or the market. The crucial properties of being a juicy sweet fruit that interacts with the pastry in a pleasing manner are shared by all apples.

Likewise, when we use letters referring to a whole set of numbers, we concentrate on properties that they all share. These are the algebraic laws, describing how the numbers behave when we do some operations on them. For instance, $a + b = b + a$, $a(b + c) = ab + ac$, and so on. In order to avoid confusion, we always have to make sure that we know exactly what sort of number can be hiding behind the letter symbol, to know what we can do with them. Then we can safely proceed and examine the consequences of their basic properties and discover new ideas.