The empowering quadratic

Attila Egri-Nagy

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The quadratic equation is powerful enough to have numerous applications in science. It also has an easy to teach general solution. Unfortunately, the appreciation of its power and beauty is often lost in education.

In presenting mathematics, there is a tendency to remove all the traces of the struggle for obtaining the results. This is due to some underlying philosophy (the platonic existence of mathematical objects vs. our fallible perception of them) and some practical considerations. The 'consumers' of the theorems may not be interested in the details.

This is not the problem of presenting theorems without proofs. A proof may or may not contain a clear indication of the thought process.

Unfortunately, this thinking diffused into education. Mathematical tests often require memorizing a formula, then substitute numerical values as required by the particular exercise. The formula is given as something to believe. This turns an exciting adventure into a humiliating drill. Solving an equation is a puzzle, it is like a detective story. But merely following some arbitrary looking rules given by someone else is not attractive at all.

Here we suggest that deriving the formula itself is a lot better exercise. Given a quadratic equation, for instance

$$x^2 + 5x + 6 = 0,$$

one can easily do a simple mental calculation to find the solutions $x = -2$ and $x = -3$. Some people just mumble something and move the fingers above the equation, then blurt out these numbers. It looks like magic, but it is actually better than that. There is a mathematical story behind. The thinking goes that $x^2 + 5x + 6$ can be written as $(x + p)(x + q)$, where p, q are new unknowns that are easy to find since

$$(x + p)(x + q) = x^2 + qx + px + pq = x^2 + (p + q)x + pq,$$

therefore $pq = 6$ and $p + q = 5$, thus $p = 2$ and $q = 3$. Finding p and q is still a quadratic problem, but in case the numbers are nice and familiar, we can do the shortcut. This is of course not general enough, since it works well with small and preferably integer numbers only. Luckily, there is a formula for the general solution.

Deriving the quadratic formula

The general quadratic equation is

$$ax^2 + bx + c = 0 \tag{1}$$

and the formula is... well, this is where the trouble begins. One has to remember the formula. It is a heap of symbols and humans are not particularly good at memorizing meaningless combinations of letters and operations. On the other hand, we can remember stories. So, what is the story behind the formula?

The power of algebra is that we can do abstract calculations, working with letters instead of numbers. What's the goal? Solving an equation is about transforming it to the form when only x is on one side. We can start working towards that goal. We tentatively separate the terms containing x from the ones without it.

$$ax^2 + bx = -c$$

There is some trouble with ax^2. In order to get x somehow we need to undo the squaring, at some point. Unfortunately, we don't know much about a. It may not be a square number. But we know something. It is not zero, so we can divide by it.

If a was zero, then we would not have a quadratic equation to start with.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

We still need to somehow undo the squaring. Now comes a bit of wishful thinking. Could we write the left side as the square of x plus something? We can easily get halfway there.

This is the method of completing the square. We know that $(x+u)^2 = x^2 + 2ux + u^2$, so we can aim to have that pattern.

$$x^2 + 2\cdot\frac{b}{2a}x = -\frac{c}{a}$$

This shows that the plus something must be $\frac{b}{2a}$. It's just that $\left(\frac{b}{2a}\right)^2$ is still missing. But this is an equation! So we can add anything to both sides!

$$x^2 + 2\cdot\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Now we have a complete square on the left side.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

We can also work on the right side a bit, doing the squaring and finding a common denominator.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{2}$$

It is time to check our progress. Equation 2 is seemingly more complicated than the starting point (1). That one at least has zero on one side. But (2) has a lot better shape. If we squint at, we can see the form $u^2 = v$. We can take the square root of both sides, getting $|u| = \sqrt{v}$. This leads to the solutions directly, but let's take a more scenic route. We can make an observation. The right side in (2) has to be non-negative, since the left side is a square number. When v is non-negative, then $v = (\sqrt{v})^2$. This may same tautological and a bit useless first, but we can transform the shape of the equation, so we have $u^2 = (\sqrt{v})^2$. Now we have squares on both sides. This again can be rearranged to $u^2 - (\sqrt{v})^2 = 0$, so the left side is easy to factor by a well-known algebraic identity.

'Squinting' means that we just look at the big picture, not the details. Algebraically this can be done by introducing new variables.

Thus not all combinations of a, b and c will give real number solutions. This is the point where we could start a different journey and construct complex numbers.

This iconic identity is one the first algebraic laws we learn: $a^2 - b^2 = (a+b)(a-b)$.

Going back to our detailed equation, we get

$$\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2 = 0,$$

which factors into

$$\left(x + \frac{b}{2a} - \sqrt{\frac{b^2-4ac}{4a^2}}\right)\left(x + \frac{b}{2a} + \sqrt{\frac{b^2-4ac}{4a^2}}\right) = 0.$$

We can make grouping of the terms a bit more explicit.

$$\left(x + \left(\frac{b}{2a} - \sqrt{\frac{b^2-4ac}{4a^2}}\right)\right)\left(x + \left(\frac{b}{2a} + \sqrt{\frac{b^2-4ac}{4a^2}}\right)\right) = 0.$$

This has the shape of $uv = 0$, so either $u = 0$ or $v = 0$. Thus, we can simply read off both solutions.

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{|2a|}$$

There seems to be a problem with $|2a|$, as $|2a| = 2a$ if $a > 0$ or $|2a| = -2a$ if $a < 0$. We may need to consider these cases. However, we already have $\pm$ for that term, so the solution set of quadratic equation is

$$S = \left\{ \frac{-b + \sqrt{b^2-4ac}}{2a}, \frac{-b - \sqrt{b^2-4ac}}{2a} \right\}.$$

That's it! Just a bit of algebraic transformations and we have a recipe for solving quadratic equations in general.

We suggest, that in addition to exercises that are mere applications of the formula, one should also do a derivation of the general solutions. It gives an empowering feeling. No need to remember the formula, it can be derived anytime.